As an example, let us take the two quadratics,

$$ax^2 + 2bxy + cy^2,$$
$$\alpha x^2 + 2\beta xy + \gamma y^2,$$

their resultant, $(a\gamma - c\alpha)^2 + 4\,(a\beta - b\alpha)\,(c\beta - b\alpha)$, belongs to the type [4: 2, 2; 2, 2] which is its own reciprocal whichever of the interchangeable elements we permute. This resultant, treating a as unity, will be equal to

$$(\alpha\rho_1^2 + 2\beta\rho_1 + \gamma)(\alpha\rho_2^2 + 2\beta\rho_2 + \gamma)$$
$$= \alpha^2\rho_1^2\rho_2^2 + 2\beta\alpha\,(\rho_1^2\rho_2 + \rho_1\rho_2^2) + 4\beta^2\rho_1\rho_2 + \alpha\gamma\,(\rho_1^2 + \rho_2^2) + 2\beta\gamma\,(\rho_1 + \rho_2) + \gamma^2$$

the image of which will be

$$\alpha^2\varepsilon_2^2 - 4\alpha\beta\varepsilon_1\varepsilon_2 + 4\beta^2\varepsilon_1^2 + 2\alpha\gamma\varepsilon_0\varepsilon_2 - 4\beta\gamma\varepsilon_0\varepsilon_1 + \gamma^2\varepsilon_0^2\,,$$

or as we may write it,

$$\alpha^2c^2 - 4\alpha\beta bc + 4\beta^2b^2 + 2\alpha\gamma ac - 4\beta\gamma ab + a^2\gamma^2\,,$$

which is $(c\alpha - 2b\beta + a\gamma)^2$, the square of the well known connective. Again, if we combine $ax^3 + 3bx^2y + 3cxy^2 + dy^3$ with $\alpha x + \beta y$, we have the invariant

$$a\beta^3 - 3b\alpha\beta^2 + 3c\alpha^2\beta - d\alpha^3\,, \quad \text{say } I,$$

belonging to the type [3 : 3, 1 ; 1, 3]. Write $\alpha = 1$, $\beta = -\rho$; this becomes

$$-a\rho^3 - 3b\rho^2 - 3c\rho - d\,,$$

of which an image, say J, belonging to the type [3 : 3, 1 ; 3, 1],

$$a\varepsilon_3 - 3b\varepsilon_2 + 3c\varepsilon_1 - d\varepsilon_0$$

is the connective of $\left\{\begin{matrix} ax^3 + 3bx^2y + 3cxy^2 + dy^3 \\ \varepsilon_0x^3 + 3\varepsilon_1x^2y + 3\varepsilon_2xy^2 + \varepsilon_3y_3. \end{matrix}\right\}$

Similarly

$$(a^2d - 3abc + 2b^3)\,\beta^3 \,\ldots\ldots + \ldots\ldots + (d^2a - 3dbc + 2c^3)\alpha^3,$$

say I, belonging to the type [6: 3, 3; 1, 3], will have for a reciprocal

$$(a^2d - 3abc + 2b^2)\,\varepsilon_3 + \ldots\ldots\,(d^2a - 3dbc + 2c^3)\,\varepsilon_0,$$

say J, belonging to the type [6: 3, 3; 3, 1]. The graph of I will be that of Fig. 41 and the graph of J, that of Fig. 42, where I use B and G (the initials of boron and gold, instead of Au for the latter) and H (the initial of hydrogen) to represent the algebraical atoms (*i. e.* quantics) of valences (*i. e.* degrees) 3, 3, and 1 respectively. Prefixing Σ to the I graph and substituting G_1, G_2, G_3, the three roots of G, for H, H', H'' and B_1, B_2, B_3 for B, B', B'' we obtain

$$\Sigma\,(B_1 - B_2)\,(B_1 - B_3)\,(B_2 - B_3)\,(B_1 - G_1)\,(B_2 - G_2)\,(B_3 - G_3),$$

which by inspection is the root representative of J, and prefixing Σ to the J graph and substituting H for G, we obtain in like manner

$$\Sigma\,(B_1 - B_2)^2\,(B_2 - B_3)\,(H - B_1)\,(H - B_3)^2,$$

as the root representative of I.

It may be observed that Fig. 43 is, algebraically speaking, a pseudograph of J, for its reading would give zero for the value of I.

It follows as an immediate consequence from the preceding extension of the law of images to quantic-systems, that the rule for deducing the first term of the reciprocal to a covariant from that of the covariant itself by writing η_r for a^r holds good as a rule for deducing each term of the one from the corresponding term of the other. To see this we have only to recall that every covariant to a quantic or quantic system may be regarded as an invariant of a new system containing the given quantic or system augmented by a linear quantic whose coëfficients are y and $-x$.

NOTE A TO APPENDIX 2.

Completion of the Theory of Principal Forms.

IN the case of a derivative from a system of k parent quantics, it at first sight would seem that since reversion (the act of forming the second image, or process, as we may term it, of double reflexion) may be effected in regard to each system of coëfficients separately, the method in the text ought to furnish in general k distinct systems of principal forms, but this is a mere mirage of the understanding which disappears as soon as the question is submitted to close examination. There is always an unique set of μ forms (μ being the multiplicity of the type) which revert unchanged (barring a numerical multiplier) whichever system of coëfficients undergoes double reflexion. But a caution is necessary for the right interpretation of this statement. $U, V, W \ldots.$ may be the principal forms in regard to one set of coëfficients, $\lambda U + \mu V, W \ldots$, or $\lambda U + \mu V + \nu W \ldots$, where λ, μ, ν are indeterminate, in regard to some other. In any such case we may still say that $U, V, W \ldots.$ is the principal system in regard to both sets and so in general. We have an example of this if we take any covariant to a single quantic Q and translate it into an invariant of Q and a linear form L. If $U, V, W \ldots$ are principal forms in respect to Q, $\lambda U + \mu V + \nu W + \ldots$ (*i. e.* the absolutely general form of the type) may be easily shown to undergo reversion in respect to L unaltered. $U, V, W \ldots.$ may consequently still be seen to be a principal form system in respect to Q and L, as each of these quantities is unaltered by reversion in respect either to Q or to L.

Suppose now a diadelphic system of which U, V are the principal forms quâ one set of coëfficients. Let R denote a reversion quâ this set, R' quâ some other set. Let $RU = aU$, $RV = bV$ and suppose $R'U = \alpha U + \beta V$. Then $R'RU = a\alpha U + b\beta V$ and $RR'U = a\alpha U + b\beta V$.

But by the nature of the process of reversion $RR' = R'R$; hence $a\beta = b\beta$. If $a = b$, every linear combination of U, V is a principal form quâ R. Hence the principal form quâ the R' set, is such for both sets. But if a is not equal to b, we must have $\beta = 0$. Hence U will be a principal form quâ R' as well as R, and the same will be true of V. For if

$$R'V = \gamma U + \delta V$$
$$RR'V = a\gamma U + b\delta V$$
$$R'RV = R'bV = b\gamma U + b\delta V.$$

Therefore $a\gamma = b\gamma$ and $\gamma = 0$. Thus U, V will each of them be common as principal forms to each set. I have gone through the same somewhat tedious process of proof for triadelphic forms and with the same result. The very beautiful conclusion follows that whatever the multiplicity of a type may be and whatever number of sets of coëfficients it involves, there is always *a single system of principal forms* common to all the sets.*

NOTE B TO APPENDIX 2.

Additional Illustrations of the Law of Reciprocity.

ACETIC aldehyde contains two atoms of carbon, one of oxygen and four of hydrogen.† It thus corresponds to the quartic covariant of a quadratic and

*Suppose there are k quantics in the parent system and that a derivative type μ is given. Each simple inversion of a pair of permutable indices (i, j) will give rise to a distinct principal equation; there will therefore be k such equations. Let ρ be a root of one of these, σ a root of any other. Then a principal form may be expressed as a linear function of any μ independent special forms connected by coëfficients which are rational integer functions of ρ. Hence σ may be found as a rational function of ρ; but in like manner ρ may be found as a rational function of σ. Hence ρ, σ must be related by an equation of the form

$$A\rho\sigma + B\rho + C\sigma + D = 0,$$

and thus we see that all the k principal equations are homographically related, *i. e.* that each may be obtained from any other by a substitution of the form

$$\rho = \frac{C\sigma + D}{A\sigma + B}.$$

In a word, the multiplicity μ (whatever the *diversity* k) determines the number of principal forms; and the k sets of principal multipliers are given by k algebraical equations of the μth degree, homographically transformable into one another.

†I originally took chloral as the subject of this investigation, being interested in examining its algebraical constitution in consequence of having had personal experience of its use as an escharotic. But for greater simplicity I have substituted acetic-aldehyde of which chloral is a third emanant, three hydrogen atoms of the former being replaced by three of chlorine in the latter.

quartic, linear and quadratic in respect to the coëfficients of the first and second respectively; such a form exists algebraically (Higher Algebra 3d ed., p. 200) and may easily be proved to be monadelphic. Let us treat it as an invariant: if we were to take for its graph a triangle of which C, C, O were the apices and attach two atoms of hydrogen to each C, the permutation-sum of the product of the differences of the connected letters is zero; this then is a pseudograph. A true graph of it is given by the figure

$$\begin{array}{c} H \cdot C \cdot O \cdot H \\ \vdots \\ H \cdot C \cdot H \end{array},$$

where each single dot between two letters means a single bond and the two dots between the upper and lower C's stand for a pair of bonds between them. This belongs to the invariantive type [4, 2; 2, 1; 1, 4: 0], the complete reciprocal to which is [2, 4; 1, 2; 4, 1: 0]. The constitution of the latter in terms of the roots is found from the above graph by writing O for C, C for H and H for O and is accordingly

$$\Sigma\,(O - O')^2\,(O - C)\,(O - O')\,(O' - C'')\,(O' - H)\,(H - C),$$

where the factor $(O - O')^2$ may be put outside the sign of summation. We may therefore take for its graph a detached molecule of oxygen + a molecule of formic acid, which latter contains two of oxygen, one of carbon and two of hydrogen.

$$\begin{array}{c} H \cdot C \cdot O \cdot H \\ \vdots \\ O \end{array}$$

will be a graph of it, from which, turning O into C, H into O and C into H we obtain

$$\Sigma\,(C - O')^2\,(C'' - H)\,(C''' - O')\,(C''' - H)\,(H - O)$$

as the value, in terms of its roots, of the algebraical equivalent to acetic aldehyde. The graph for formic acid, it may be noticed, exists algebraically (Higher Algebra, p. 300).

Instead of the dissociated molecules of oxygen and formic acid, we may exhibit them combined in the graph

$$\begin{array}{l} C \cdot O \cdot O \cdot O \cdot H \\ \vdots \\ O \end{array}$$

which will give another form to the value of the reciprocal in question, viz.

$$\Sigma\,(C - H)^2 (H - O)\,(H - C')\,(C' - C'')\,(C'' - C''')\,(C''' - O)$$

which, not being zero and the type being monadelphic,* must be in a pure numerical ratio to the sum above written.

Chemistry has the same quickening and suggestive influence upon the algebraist as a visit to the Royal Academy, or the old masters may be supposed to have on a Browning or a Tennyson. Indeed it seems to me that an exact homology exists between painting and poetry on the one hand and modern chemistry and modern algebra on the other. In poetry and algebra we have the pure idea elaborated and expressed through the vehicle of language, in painting and chemistry the idea enveloped in matter, depending in part on manual processes and the resources of art for its due manifestation.

A peculiar case might possibly arise in applying the theory of principal forms to a self-reciprocal type $[w : i, i]$ which it is proper to mention. For greater simplicity suppose the type to be diadelphic and let M, N be forms of the type which satisfy the equations

$$IM = \rho M\,, \quad IN = \rho' N\,;$$

the M and N have tacitly been defined to be the principal forms for such a type. Now in general this definition merges into and is coincident with the definition of principal forms for the general case, viz., that I^2M and I^2N must be multiples of M and N and the latter condition might be substituted for the former. But this is not always true, for if $\rho + \rho' = 0$, we shall have

$$I^2M = \rho^2 M\,, \quad I^2N = \rho^2 N\,,$$

and consequently,

$$I^2(M + \lambda N) = \rho^2(M + \lambda N)\,,$$

so that if we were to follow the general definition the principal forms might become indeterminate, whereas by following the definition special to the self-reciprocal case they are determinate. Thus ex. gr., suppose that P, Q, two particular forms of the type, satisfy the equations

$$IP = \rho Q\,, \quad IQ = \sigma P\,;$$

* As an exercise the reader may satisfy himself that this type is monadelphic by the direct application of the rule for finding the multiplicity. It corresponds to a quadratic covariant of the type $[2, 4; 4, 1: 2]$, which is the same (introducing the weight $\frac{2 \,.\, 4 + 4 \,.\, 1 - 2}{2}$ in lieu of the degree) as the type $[5: 2, 4; 4, 1]$ and has the same multiplicity μ by the law of reciprocity as the type $[5: 4, 2; 4, 1]$, viz. the difference between the number of modes of composing 5 and of composiog 4 with two of the numbers 0, 1, 2, 3, 4 and with one of a *distinct* set of the same numbers. The arrangements for the weight 5 will be 4. 1 : 0, 4. 0: 1, 3. 2: 0, 3. 1 : 1, 3. 0 : 2, 2. 2 : 1, 2. 1 : 2, 2. 0 : 3, 1. 1 : 3, 1. 0 : 4, and for the weight 4, 4. 0 : 0, 3. 1 : 0, 3. 0: 1, 2. 2 : 0, 2. 1: 1, 2. 0: 2, 1. 1 : 2, 1. 0 : 3, 0. 0 : 4. The numbers of the combinations in the two sets of arrangements are respectively 10 and 9. Hence $\mu = 10 - 9 = 1$, or the type is monadelphic. The same result of course follows from the known fundamental scale for a quadro-biquadratic system.

the principal forms will then be

$$\sqrt{\sigma}\,P + \sqrt{\rho}\,Q \text{ and } \sqrt{\sigma}\,P - \sqrt{\rho}\,Q\,,$$

and the two principal multipliers become $\sqrt{\rho\sigma}$ and $-\sqrt{\rho\sigma}$, so that the principal forms according to the general definition would be indeterminate, but according to the definition proper to self-reciprocal forms strictly determinate.

Let us, as a final example of self-reciprocal type, consider the type [10: 5, 5] which is the same as [5, 5: 5] and corresponds to the covariant of the 5th order in the coëfficients and of the 5th degree in the variables to a quintic. This is diadelphic, as may be found by consulting the table of irreducible forms for the quintic, which will show that it can arise only from the multiplication of the parent quantic itself by its quartinvariant or from that of the quadratic quadri-covariant by the cubic cubo-covariant or from a linear combination of the two products. But without this, the same conclusion may be arrived at by direct calculation of the value of (10: 5, 5) — (9: 5, 5) and the multiplicity will be found to be 18 — 16, or 2 as premised. Let us take as our special forms,

$$P = (ae - 4bd + 3c^2)\,(ace + 2bcd - ad^2 - c^3 - b^2e),$$

$$Q = a\,(a^2f^2 - 10abef + 4acdf + 16ace^2 - 12ad^2e + 16b^2df + 9b^2e^2 - 12bc^2f - 76\,bcde + 48\,bd^3 + 48c^3e - 32c^2d^2),$$

where $\frac{Q}{a}$ is the quartinvariant J given by Salmon, p. 207, (3d ed.), being in fact the discriminant of the quadricovariant whose root-differentiant is $ae - 4bd + 3c^2$. Call $\alpha, \beta, \gamma, \delta, \varepsilon$ the five roots of the quintic and make $a = 1$. Q contains the term f^2 which is the image of $\alpha^5\beta^5$ which can only arise from combinations of the coëfficients into which d, e, f none of them enter. But all the terms of Q contain d, e, or f, moreover P has no term containing f^2, therefore IQ does not contain Q but is simply a multiple of P. Again ce^2, which enters into P, is the image of combinations of the form $\alpha^2\beta^4\gamma^4$, and the only term in Q which can give rise to such combinations is $-32c^2d^2$, or

$$-\frac{32}{10^4}\,(\Sigma\alpha\beta)^2\,(\Sigma\alpha\beta\gamma)^2\,,$$

and each such combination will have unity for its coëfficient and their number is 30. Hence

$$IQ = -\frac{30\cdot 32}{10000}\,P = -\frac{12}{125}\,P\,.$$

Again, Q contains $-10bef$, and bef is the image of such root-combinations as $\alpha^5\beta^4\gamma$ (60 in number) the only terms in P capable of producing which are $10bc^3d$ and $-3c^5$ or $\frac{1}{5000}\,\Sigma\alpha\,(\Sigma\alpha\beta)^3\,\Sigma\alpha\beta\gamma \;-\; \frac{3}{100000}(\Sigma\alpha\beta)^5$. And bef does not appear in P, hence one part of IP will be

$$\left(\frac{60}{-50000}+\frac{3\cdot5\cdot60}{1000000}\right)Q\,,\quad\text{or}\;-\frac{3}{10000}\,Q\,.$$

Again, ce^2 is the image of such combinations as $\alpha^4\beta^4\gamma^2$ (30 in number) and the only terms in P giving rise to such are $-3c^5-8b^2cd^2+10bc^3d-3c^2d^2$; $-3c^5$ is $-\frac{3}{100000}\,(\Sigma\alpha\beta)^5$ and will give rise to $-\frac{3\cdot20\;30}{100000}\,ce^2$ in IP; $-8b^2cd^2$ is $-\frac{8}{25000}\,(\Sigma\alpha)^2\,(\Sigma\alpha\beta)\,(\Sigma\alpha\beta\gamma)^2$ and will give rise to $-\frac{2\cdot8\cdot30}{25000}\,ce^2$ in IP; $10bc^3d$ is $\frac{10}{50000}\,\Sigma\alpha\,(\Sigma\alpha\beta)^3\,\Sigma\alpha\beta\gamma$ and will give rise to $\frac{7\cdot10\;30}{50000}\,ce^2$ in IP; $-\;3c^2d^2$ is $-\frac{3}{10000}\,(\Sigma\alpha\beta)^2.(\Sigma\alpha\beta\gamma)^2$ and will give rise to $-\frac{3\cdot30}{10000}\,ce^2$ in IP. Hence the total coëfficient of ce^2 in IP is

$$-\frac{9}{500}-\frac{12}{625}+\frac{21}{500}-\frac{9}{1000}=\frac{-90-96+210-45}{5000}=-\frac{21}{5000}\,,$$

and consequently, since P contains the term ce^2 and Q the term $16ce^2$, if $IP=\theta P-\frac{3}{10000}\,Q$,

$$\theta-\frac{3\cdot16}{10000}=-\frac{21}{5000}\,,\;\text{so that}\;\theta=\frac{3}{5000}\,,$$

and therefore

$$IP=\frac{3}{5000}\,P-\frac{3}{10000}\,Q\,,$$

and thus the equation for finding the principal multipliers ρ is

$$\begin{vmatrix}\frac{3}{5000}-\rho, & -\frac{3}{10000}\\ -\frac{12}{125}\,, & -\rho\end{vmatrix}=0,$$

or, if $\rho=\frac{3\sigma}{10000}\,,\quad \begin{vmatrix}2-\sigma\,, & -1\\ -320, & -\sigma\end{vmatrix}=0.$

Thus $\sigma^2 - 2\sigma - 320 = 0$, the roots of which are irrational. I have thought it advisable to set out the work in this example with some explicitness in order to remove an impression that might otherwise arise from the examples which precede, that the principal multipliers and consequently the principal forms, for self-reciprocal types, necessarily contain only rational numbers.

The work is very much longer for the case of non-self-reciprocal types. The simplest example of such that presents itself to my mind is that of the sextinvariant of a quartic and the quartinvariant of a sextic, for either of which the type is diadelphic. The discussion of this case forms the subject of the annexed Note, for all the calculations of which I am indebted to the labor and skill of Mr. F. Franklin, Fellow of Johns Hopkins University. For the sake of brevity the steps of the work have been suppressed and only the final results set out.

NOTE C TO APPENDIX 2.

On the Principal Forms of the General Sextinvariant to a Quartic and Quartinvariant to a Sextic.

Let

$$L = (ae - 4bd + 3c^2)^3 = \left[\frac{1}{2^3\cdot 3}\Sigma(\alpha-\beta)^2(\gamma-\delta)^2\right]^3,$$

$$M = \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}^2 = (ace + 2bcd - ad^2 - b^2e - c^3)^2 = \left[\frac{1}{2^4\cdot 3^3}\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(\alpha-\gamma)(\beta-\delta)\right]^2,$$

$$P = (ag - 6bf + 15ce - 10d^2)^2 = \left[-\frac{1}{2^4\cdot 3\cdot 5}\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(\varepsilon-\phi)^2\right]^2,$$

$$Q^* = \begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix} = \begin{cases} aceg - acf^2 - ad^2g + 2adef \\ - ae^3 - b^2eg + b^2f^2 + 2bcdg \\ - 2bcef - 2bd^2f + 2bde^2 - c^3g \\ + 2c^2df + c^2e^2 - 3cd^2e + d^4 \end{cases}$$

$$= \frac{1}{2^5\cdot 3^3\cdot 5^3}\Sigma(\alpha-\beta)^4(\gamma-\delta)^4(\varepsilon-\phi)^4 - \frac{71}{2^{10}\cdot 3^4\cdot 5^4}\left[\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(\varepsilon-\phi)^2\right]^2.$$

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*M. Faà de Bruno, in the tables at the end of his "Théorie des Formes Binaires," designates Q and $\Sigma(\alpha-\beta)^4(\gamma-\delta)^4(\varepsilon-\phi)^4$ by the same symbol I_4; a misleading circumstance which gave rise in this instance, and might in others to a large amount of useless labor. As can easily be seen from the above, the true value of $\Sigma(\alpha-\beta)^4(\gamma-\delta)^4(\varepsilon-\phi)^4$ is $120\,(71P + 900\,Q)$
$= 120\,(71a^2g^2 - 852abfg + 3030aceg - 900b^2eg - 2320ad^2g + 1800bcdg - 900c^3g - 900acf^2 + 3456b^2f^2 + 1800adef - 14580bcef + 6720bd^2f + 1800c^2df - 900ae^3 + 1800bde^2 + 16875c^2e^2 - 24000cd^2e + 8000d^4)$. It should also be observed that in the expression for Q (the catalecticant) given in the same table, the signs of the terms $-2bd^2f + 2bde^2$ have been interchanged.

Then

$$IL = \frac{P - 6Q}{2^5 \cdot 3^2}, \qquad IM = \frac{P - 33Q}{6^5},$$

$$IP = \frac{L + 2M}{2^4 \cdot 5}, \qquad IQ = \frac{9L - 142M}{2^6 \cdot 3^2 \cdot 5^3},$$

$$I^2 L = \frac{7614L + 23868M}{2^{11} \cdot 3^6 \cdot 5^3}, \qquad I^2 M = \frac{201L + 2162M}{2^{11} \cdot 3^6 \cdot 5^3}.$$

In order that $\lambda L + \mu M$ shall be a principal form we must have

$$(7614 - 2^{11} \cdot 3^6 \cdot 5^3 \rho)\, \lambda + 201\mu = 0,$$

$$23868\lambda + (2162 - 2^{11} \cdot 3^6 \cdot 5^3 \rho)\, \mu = 0,$$

$$\begin{vmatrix} 7614 - 2^{11} \cdot 3^6 \cdot 5^3 \rho, & 201 \\ 23868, & 2162 - 2^{11} \cdot 3^6 \cdot 5^3 \rho \end{vmatrix} = 0,$$

or, putting $\sigma = 2^8 \cdot 3^6 \cdot 5^3 \rho$,

$$\sigma^2 - 1222\sigma + 182250 = 0,$$

where it may perhaps be worth noticing that the last term is $2 \cdot 3^6 \cdot 5^3$ and the coëfficient of the second term $2\ 13 \cdot 47$. We obtain from this equation

$$\rho = \frac{611 \pm \sqrt{191071}}{2^8 \cdot 3^6 \cdot 5^3}.*$$

The principal forms in L and M will then be found to be

$$201L + (-2726 + 8\sqrt{191071})\, M, \qquad 201L + (-2726 - 8\sqrt{191071})\, M;$$

and those in P and Q

$$101P + (-11436 + 24\sqrt{191071})\, Q, \qquad 101P + (-11436 - 24\sqrt{191071})\, Q.$$

Or, if we please, the principal forms in the two cases may be taken as the factors of $201L^2 - 5452LM - 23868M^2$ and $101P^2 - 22872PQ + 205200Q^2$ respectively.† The question, what reduced quadratic forms can appear in the theory of diadelphic types, may one day or another become the subject of *à priori* investigation and form a new connecting link between the Calculus of Invariants and the Theory of Numbers. The linear functions of L and M and of P and Q, corresponding to the reduced forms of the above expres-

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* The number under the radical sign is, I believe, a prime number, but I have not within reach the tables necessary for verifying this. Professor Newcomb, by an exceedingly ingenious combination of a table of squares with Crelle's table of multipliers, (a real stroke of genius,) was able to ascertain by an inspection (the work of a few minutes) that 191071, if not a prime number, must contain a factor not greater than a certain moderate sized integer (137 if my memory serves me right) which reduces the trials necessary to be made to a very small compass.

† These are reducible to

$(201, 68, -60800 \between L', M)^2$, $(101, -23, -1089667 \between P', Q)^2$, where $L' = L - 14M$, $P' = P - 113Q$.

sions might perhaps be termed the principal *rational* forms of the two types respectively.

It may be well to notice that if $I^2U = \rho U$, then $I^2 \cdot IU = I \cdot I^2 U = \rho IU$, and consequently the principal forms for two reciprocal types are images respectively of one another, and the principal multipliers are the same for the two systems.

NOTE D TO APPENDIX 2.

On the Probable Relation of the Skew Invariants of Binary Quintics and Sextics to one another and to the Skew Invariant of the same Weight of the Binary Nonic.

THE law of reciprocity extended, as it has already been in these pages, to systems of quantics, admits of an additional important generalization.

We know that Regnault's law of substitution holds good for algebraical forms, and in fact when transferred to the algebraical sphere becomes identical with the method which I believe I was the first to employ (now familiar to algebraists through the use made of it by Professors Clebsch and Gordan) to which I gave the name of emanation, (Faà de Bruno, p. 198).

The principle, stated in chemico-algebraical language, is that in algebraical compound any number of atoms of a given valence may be replaced by the same number of *new* equi-valent atoms. [In algebra it is essential to lay a peculiar stress on the word *new*; for if the substituted atoms should be homonymous with the remaining atoms, there is a possibility of the transformed compound reducing to zero. As for instance in the algebraical compound $ab' - a'b$ (the representative, say, of potassic iodide), if the atom of potassium should be changed into another of iodine, (or *vice versa*), the compound, viewed algebraically, would disappear].

The law of reciprocity as I have previously given it, translated into chemico-algebraical language amounts to saying that the total number of atoms of one kind (say m n-valent of one kind) may be replaced by n m-valent atoms of another kind; but by applying the rule of substitution first and then that of reciprocity we may see that the condition of *totality* may be done away with and the proposition reduced to the simplified form that in any algebraical compound *m n-valent atoms may be replaced by n m-valent ones.* Whether this law has any application in the chemical sphere, I must leave to chemists to determine.

In addition to the well known fact that a quintic possesses an invariant of the 18th order and a sextic, one of the 15th order, having obtained a complete scheme of the irreducible invariants for the binary quantic of the 10th degree, I was put in possession of the new fact that this last form possesses an invariant of the 9th order and consequently that the nonic possesses an invariant of the 10th order.*

Now the weight of each of these skew invariants is the same number 45, and I was thus led to suspect that they coëxisted in virtue of some secret connexion. What that connexion is I think that I am now (very unexpectedly) in a position to explain and to show (with a high degree of probability) how the values of these three invariants may be actually deduced and calculated from one another. This follows as a consequence of the combined laws of reciprocity and substitution otherwise called emanation. For suppose we have an invariant of a quantic of the mth degree, of the order np in the coëfficients. By the principle of emanation we may transform this into an invariant to a system of n quantics, each of the degree m and of the order p in each set of coëfficients, and by the generalized law of reciprocity this may be again transformed into an invariant to a system of n quantics, each of degree p and of the order m in each set of coëfficients. If now finally these n quan-

*I have calculated, with the kind assistance of Mr. Halsted, the expression in its canonical form of the generating fraction to a binary quantic of the 10th degree. The coëfficient of m in this fraction developed, represents the number of parameters in the general invariant of the mth order of the given decadic. Its denominator is

$$(1-t^2)(1-t^4)(1-t^6)^2(1-t^8)\,1-t^9)\,1-t^{10})\,1-t^{14})$$

and its numerator is the rational integer function

$$1+2t^6+\ldots\ldots\quad+2t^{42}+t^{48},$$

the successive coëfficients being
1, 0, 0, 0, 0, 0, 2, 0, 4, 2, 7, 6, 15, 13, 16, 25, 22, 31, 34, 40, 41, 47, 46, 49, 48, 49, 46, 47, 41, 40, 34, 31, 22, 25, 16, 13, 15, 6, 7, 2, 4, 0, 2, 0, 0, 0, 0, 0, 1,
showing that the primary fundamental invariants are of the orders 2, 4, 6, 6, 8, 9, 10, 14, and that (by the law of "Tamisage" *anglice* siftage) the secondary (or as they might be better termed the auxiliary) ones are of the orders 6, 8, 9, 10, 11, 12, 13, 14, 15, 17 taken 2, 4, 2, 7, 6, 12, 13, 18, 21, 11 times respectively. Any other invariant of the decadic can be represented as a linear function of a limited number of combinations of the secondaries, having for its coëfficients some combination of powers of the primaries.

Suppose that the same numerical order occurs among the primaries and secondaries, as ex. gr. 6, which occurs twice among the former and twice among the latter. This will indicate in the first place that, calling A and B the quadric and quartic invariants, the general sextic one will be of the form

$$\lambda A^3+\mu AB+\nu_1 Q_1+\nu_2 Q_2+\nu_3 Q_3+\nu_4 Q_4$$

and that any two independent special values of $\nu_1 Q_1+\nu_2 Q_2+\nu_3 Q_3+\nu_4 Q_4$ may be taken as primaries and any other independent two as secondaries, and so in general; I mention this to prevent the false suggestion, which might otherwise arise, that the secondaries and primaries are different in internal constitution. This remark receives a beautiful illustration in an algebraical theory (recently developed by me) of chemical isomerism, which gives rise to a generating function precisely similar in character to that applicable to in- and covariants and is subject to a similar law of interpretation, graphs taking the place of algebraical forms, and atomicules and the numbers of grouped atoms, of degrees and orders.

tics, be all made identical with one another, then the transformed invariant, *provided it does not vanish*, becomes an invariant of the order mn to a single quantic of the degree p, and accordingly we may pass in certain cases from the type $[m, np: 0]$ to the type $[p, mn: 0]$. So in all probability we may pass from the type $[5, 18: 0]$ to the type $[6, 15: 0]$ and to the type $[9, 10: 0]$. As there is only one invariant of the type $[6, 15: 0]$, or of the type $[9, 10: 0]$, it follows that, if the passage from type to type is real and not nugatory, the three invariants of these second types may be deduced, any one from any other, by the explicit processes above described. There is nothing at all doubtful in the course of the transformation except what arises from the possibility that in the last step of it the effect of rendering identical the different sets of coëfficients—*i. e.* of finding the counter-emanant, so to say, of the invariant containing n sets of variables—may be to render the whole expression null. This of course would happen if we attempted to pass from the type $[5, 18: 0]$ to the type $[3, 30: 0]$, or to the type $[2, 45: 0]$, which we know are void of forms. But there is no reason why we should expect this to happen when we pass from the given type to other types known to contain one or more forms. It would require no impracticable amount of labor to actually verify the fact of the transformation being effectual between the skew invariants of the sextic and quintic forms. The survival of a *single* known term in either of them, in the process of attempting to deduce it from the other, would be sufficient to establish the effectualness of the method, and that it will be found to be effectual, for reasons too long to dwell upon here, I scarcely entertain a doubt. The process to be employed may be summarily comprehended under the three rubrics of diversification, reciprocation and unification. The first is one of differentiation alone; the second involves the expansion of functions of the coëfficients of an equation in terms of roots and the substitution of η_i for α^i; the third consists merely in replacing distinct sets of letters (η) by a single set. In practice the two latter processes would be of course combined into one. It will be instructive to consider some simple example of this method of transformation of types.

Let us take $(ac - b^2)^3$ regarded as belonging to the type $[2, 6: 0]$. I shall show how to pass from this to a form of the type $[3, 4: 0]$. Taking a third emanant of the given form, *i. e.* the result of the operation upon it of $\dfrac{1}{1\cdot 2\cdot 3}(a'\delta_a + b'\delta_a)^3$, we obtain

$$(ac' + a'c - 2bb')^3 + 2(ac - b^2)(a'b' - b'^2)(ac' + a'c - 2bb').$$

Let us call α, β, α', β' the roots of the two forms $[1, b, c]$, $[1, b', c']$ respectively; then the emanant last found (multiplied by 8) becomes

$$(2\alpha\beta + 2\alpha'\beta' - \alpha\alpha' - \alpha\beta' - \beta\alpha' - \beta\beta')(\overline{2\alpha\beta + 2\alpha'\beta' - \alpha\alpha' - \alpha\beta' - \beta\alpha' - \beta\beta'}^2 + \overline{\alpha - \beta}^2 \cdot \overline{\alpha' - \beta'}^2) .$$

After performing all the multiplications and introducing the zero powers of α, α', β, β' in such terms as do not contain one or more of these letters, all that remains is to substitute

$$\begin{aligned} \alpha^0 &= \alpha'^0 = \beta^0 = \beta'^0 = \quad a, \\ \alpha &= \alpha' = \beta = \beta' = -b, \\ \alpha^2 &= \alpha'^2 = \beta^2 = \beta'^2 = \quad c, \\ \alpha^3 &= \alpha'^3 = \beta^3 = \beta'^3 = -d, \end{aligned}$$

the letters a, b, c, d for greater simplicity being used instead of ε_0, ε_1, ε_2, ε_3, *i. e.* η_0, $-\eta_1$, η_2, $-\eta_3$. The result will not vanish. To show this consider the group of terms which change into a^2d^2. These are the binary combinations of α^3, α'^3, β^3, β'^3. $2\alpha\beta$ and $2\alpha'\beta'$ in the first factor give rise to $8\alpha^3\beta^3$, $8\alpha'^3\beta'^3$ and the remaining four terms to $-2\alpha^3\alpha'^3$, $-2\alpha^3\beta'^3$, $-2\beta^3\alpha'^3$, $-2\beta^3\beta'^3$ respectively. Hence the term a^2d^2 will survive with the multiplier $8 + 8 - 2 - 2 - 2 - 2$, *i. e.*, 8. So again the only terms introducing ac^3 will be the ternary combinations of α^2, α'^2, β^2, β'^2. $2\alpha\beta$ and $2\alpha'\beta'$ will be found to produce as many positive as negative terms of this kind, but $-\alpha\alpha'$ will produce $4\alpha^2\alpha'^2\beta^2 + 4\alpha^2\beta'^2\beta'^2$, giving rise to $8ac^3$, and as the same will be true for $-\alpha\beta'$, $-\beta\alpha'$, $-\beta\beta'$, we see that $32ac^3$ will emerge in the result. Hence the given invariant becomes converted into

$$(a^2d^2 + 4ac^3 + \ldots\ldots) ,$$

i. e. the discriminant of the cubic whose type is $[3, 4: 0]$ as was to be shown.

I think it is little doubtful that wherever there exist forms contained under each of two types, the product of whose rank and order is identical, we may pass from the one to the other by means of the combined processes of emanation and reciprocation, as in the foregoing example.* [The case is

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* Call $(b^2 - ac)^3 = A$, $a^2d^2 + 4ac^3 + \ldots\ldots = B$, $a'\delta_a + b'\delta_b + c'\delta_c = E$, $a\delta_{a'} + b\delta_{b'} + c\delta_{c'} + d\delta_{c'} = H^{-1}$. Then it follows from the text that

$$B = \tfrac{1}{12}H^{-2}IE^3A ,$$

where it may be observed that E^3A is diadelphic, for it will be proved that $(6: 3, 2; 3, 2) = 16$, and $(5: 3, 2; 3, 2) = 14$, so that any form whatever coming under the same type as E^3A is a linear function of $(ac' + a'c - 2bb')^3$ and $(ac' - a'c - 2bb')(ac - b^2)(a'c' - b'^2)$, say L and M, (whose difference, $L - M$, is $\frac{1}{6}E^3A$), and operated on by $H^{-2}I$ would produce a multiple of B (whose type is monadelphic) with the sole exception of $\lambda L - 2\lambda M$, the result of operating upon which would be zero. Similarly we may see that in any given case the chances are infinitely in favor of the expectation that the process will *not* be nugatory by which it has been shown we may pass from one known type $[m, np: 0]$ to another known one $[p, nm: 0]$.

much the same as with transvection. That process may produce a null form, but any actually existent form may be produced by it and exhibited as a transvect.] To pass from Hermite's to Cayley's skew form, we must first by emanation change [5, 18: 0] into [5, 6; 5, 6; 5, 6: 0] and then this latter into [6, 15, 0]; by means of the process last exemplified.

APPENDIX 3.

ON CLEBSCH'S THEORY OF THE "EINFACHSTES SYSTEM ASSOCIIRTER FORMEN" (*vide Binären Formen, p. 330*) AND ITS GENERALIZATION.

LET $(a, b, c, \ldots . k, l \between x, y)^n$ be any binary quantic. Let the provector symbol $(l\delta_k + 2k\delta_h + 3h\delta_g + \ldots .)$ be denoted by Ω, and the revector symbol $(a\delta_b + 2b\delta_c + 3c\delta_d + \ldots .)$ by $\mho$. Let Q_{2i} represent the quadrinvariant of the above form when $n = 2i$. Now let Ω and $\mho$ be made to comprise the $2i + 1$ letters $a, b, c, \ldots . l, m$; then $a\Omega Q_{2i} - 2bQ_{2i}$* will be nullified by the operation of $\mho$ and will therefore be a cubinvariant for the case of $n = 2i + 1$ which we may call Q_{2i+1}. Also let $Q_0 = a$; then $Q_0, Q_1, Q_2, \ldots . Q_\mu$ will be differentiants to all binary quantics of degree equal to or greater than μ. The above I call basic differentiants. Their distinguishing characteristic is that the highest letter in each of them enters into it only in the first degree multiplied by a or by a^2 and by no other letter. Now let D be any given differentiant of degree μ and for the moment make $a = 1$. Then it is obvious that D may be expressed—by means of successive substitutions of its ultimate, its penultimate, its antepenultimate, etc. letters up to c inclusive, in terms of the corresponding basic differentiants and the anterior letters,—as a rational integer function of $Q_1, Q_2, \ldots . Q_\mu, b$; or, restoring to a its general value, will be a rational integer function of $Q_0, Q_1, Q_2, \ldots . Q_\mu, b$, say F, divided by a power of a. But I say that b will have disappeared in the process. For $\mho D = 0$; and $\mho Q_0 = 0$, $\mho Q_1 = 0 \ldots . \mho Q_\mu = 0$. Hence, regarding each Q as a constant, $\left(a\frac{d}{db}\right)F = 0$, or F does not contain b.

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*For by a well known formula if D is a differentiant in x of the type $[w : i, j]$, $\mho\Omega D = (ij - 2w)\,D$. Consequently when Q_{2i} is regarded as a differentiant in x of the type $[2i : 2i + 1, 2]$ $\mho\Omega Q_{2i} = Q_{2i}$ also $\mho Q_{2i} = 0$ and $\mho b = a$. Hence $\mho\,(a\Omega Q_{2i} - 2bQ_{2i}) = 0$.

Again, suppose we take a system of two quantics and let $Q_0, Q_1, \ldots . Q_\mu$ be the basic differentiants of the one, $Q'_0, Q'_1, \ldots . Q'_\nu$ of the other, and let D be any differentiant of the system. Then by the same method as before we shall find

$$D = \frac{F(Q_0, Q_2 \ldots . Q_\mu : Q'_0, Q'_2 \ldots . Q'_\nu : b, b')}{a^m \cdot a'^n}.$$

Also each Q will be nullified by $\mho$, and each Q' by $\mho'$, and therefore each Q and Q' as well as D will be nullified by the operator $\mho + \mho'$. Hence we shall have

$$\left(a \frac{d}{db} + a' \frac{d}{db'}\right) F = 0,$$

or

$$F = \phi \, (ab' - a'b) ,$$

ϕ being a rational integral form of function. In like manner for a system of three quantics, regarding the several sets of its basic differentiants as constant, we shall have

$$F = \phi \, (ab' - a'b : ac' - a'c : bc' - b'c) ,$$

where ϕ is a rational integral form of function, or

$$F = \psi \, (ab' - a'b : ac' - a'c : a, a') ,$$

and so in general. Hence, remembering that any relation between differentiants must continue to subsist between the covariants of which they are the roots, and now, understanding by base forms the complete covariants of which the basic coëfficients are the roots, we may pass from differentiants to in- or co-variants and obtain the following theorems.

1°. For a single quantic of degree i, any in- or co-variant is expressible by a fraction whose numerator is a rational integer function of its i base forms and whose denominator is a power of the quantic. This is Clebsch's theorem.

2°. For a system of quantics, any in- or co-variant is expressible by a fraction whose numerator is a rational integer function of the separate base forms of its several quantics and of any complete system of $(\mu - 1)$ independent Jacobians of the quantics taken in pairs, and whose denominator is a product of powers of the quantics of the system.

Also it will be observed that these theorems will continue to subsist when the base forms have for their roots in lieu of the basic differentiants, as above

defined, any ascending scale of differentiants in which the letters enter successively one at a time and each letter on its first appearance figures only in the first degree and combined exclusively with powers of a.

On the theory of basic forms may be grounded a method for obtaining, *in propriâ personâ*, the fundamental in- and co-variants to a quantic or system of quantics in regular succession, by a process which continues so long as there are many more to be elicited and comes to a self-manifesting end as soon as the last irreducible form has been obtained, like an air pump that refuses to act as soon as the exhaustion has become complete. In a word, the cataloguing of the irreducible in- and co-variants is transferred to the province of, and becomes a problem in, ordinary algebra.

I have previously observed that any expression which represents a differentiant in regard to a quantic of a given degree necessarily does the same for quantics of all higher degrees. And I may take this occasion to remark, or to repeat, that a differentiant may be irreducible in respect to the quantic of minimum degree to which it can be referred, and yet not so for quantics of higher degrees. Thus, if we take the expression

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd,$$

this referred to a cubic is irreducible (as is well known), but regarded as a differentiant of a quartic or higher degreed quantic, is reducible, being in fact identical with

$$(ac - b^2)(ae - 4bd + 3c^2) - a\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}.$$

Let us suppose a linear function $yu - xv$ combined with a quantic into a system. Then it follows as a corollary from (2° at p. 118), that if the quantic belongs to the form $(a, b, c, \ldots . l \between u, v)^i$, or say more simply to the form $[a, b, c, \ldots . l]$ any covariant of such quantic multiplied by a suitable power of a will be a function of y, $ax + by$ and of the differentiants, or in a word, every covariant of the quantic expressed as a function of x and $ax + by$ will have no coëfficients but what are differentiants, or to use Professor Cayley's term, semi-invariants. Thus, ex. gr., the Hessian of the cubic $(a, b, c, d \between x, y)^3$ may be put under the form

$$\frac{1}{a^2}\left\{(ac - b^2)(ax + by)^2 + (a^2d - 3abc + 2b^3)(ax + by)y + (ac - b^2)^2y^2\right\},$$

So it will be found that the Hessian of the quintic, viz.

$$(ae - 4bc + 3c^2)\,x^2 + (af - 3be + 2cd)\,xy + (bf - 4cd + 3d^2)\,y^2$$

on writing $ax + by = X$ becomes

$$\frac{1}{a^2}\left\{ (ae - 4bc + 3c^2)\,X^2 + (a^2f - 5abe + 2acd + 8b^2d - 6bc^2)\,Xy \right.$$
$$\left. - \Big[(ac - b^2)(ae - 4bd + 3c^2) + 3a\,(ace + 2bcd - ad^2 - b^2e - c^3)\Big]\,y^2 \right\},$$

where all the coefficients are semi-invariants-in-x, the second coefficient being one of the basic differentiants and the latter part of the third coefficient, the catalecticant

$$\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix},$$

and so more generally, it may be shown to follow from (2°), that if there be any number of binary quantics

$$[a, b, c \ldots .]\,,\ [a', b', c', \ldots .]\,,\ [a'', b'', c'' \ldots .]\,,$$

every covariant of such system, expressed as a function of y and of *any one* of the quantics

$$ax + by\,,\ a'x + b'y\,,\ \ldots\ldots$$

chosen at will, has differentiants-in-x exclusively for its coefficients.

It is easy to express the base-covariants in terms of the roots. Those of weight $2n$ and order 2 will be of the form

$$\Sigma F\,(a_1, a_2, a_3, \ldots .\, a_{2n})\,(x - a_{2n+1})^2 (x - a_{2n+2})^2 \ldots\ldots$$

where F may be expressed as

$$(a_1 - a_2)^2 (a_3 - a_4)^2 \ldots\ldots (a_{2n-1} - a_{2n})^2\,,$$

or, $$(a_1 - a_2)(a_2 - a_3)(a_3 - a_4) \ldots\ldots (a_{2n-1} - a_{2n})(a_{2n} - a_1)\,,$$

or under a variety of other forms all equal to a numerical factor près; for the type $[2n : 2n, 2]$ and the more general one $[2n : 2n + \nu, 2]$ are monadelphic. And again those of the weight $2n + 1$ and order 3 may take, or at all events be replaced by, the form

$$\Sigma\left(\overline{a_1 - a_2}\,.\,\overline{a_2 - a_3} \ldots \overline{a_{2n-1} - a_{2n}}\,.\,\overline{a_{2n} - a_1}\,.\,\overline{a_1 - a_{2n+1}}\,.\,\overline{x - a_1}\,.\,\overline{x - a_2} \ldots \overline{x - a_{2n+1}}\,.\,\overline{x - a_{2n+2}}^{\,3}\,.\,\overline{x - a_{2n+3}}^{\,3} \ldots\right)$$

It is proper to notice that the type $[2n + 1 : 2n + 1 + \nu\,;\, 3]$ is only monadelphic so long as $2n + 1$ is less than 9, so that we cannot, without an investigation which might be tedious, determine whether the above representation coincides with the basic forms of the third order in the coefficients adopted in p. 118; but such investigation would be a work of supererogation, for the only *material* character for any of the base-covariants in question to possess

is, that its root differentiant-in-x shall be not higher than of the third order in the coefficients and shall contain the element ε_{2n+1}. Any formula having this property (which is enjoyed by the root function above given) is just as good as any other for the purposes of this theory.*

It will be seen to follow from the theorem I have given for differentiants from which Clebsch's follows as an immediate consequence, that all the permutation-sums of any rational integer function of the differences of the roots of an algebraical equation of the nth degree are rational integer functions of $(n-1)$ of them of the second and third order alternately; so, for example, all the coefficients in Lagrange's equations to the squares of the differences of the roots of an algebraical equation in its ordinary form are rational integer functions of $(n-1)$ known quantities. Thus, for instance, the equation to the squares of the differences of a cubic equation will be

$$\rho^3 + 18(b^2 - ac)\rho^2 + 81(b^2 - ac)^2 + 27\Delta = 0,$$

where the coefficients are given in terms of two differentiants $(b^2 - ac)$ and Δ.

Throughout this paper the perspicuity of expression has been considerably marred by want of a complete nomenclature which the theory of graphs and types necessarily calls for and which I shall hereafter employ whenever I may have occasion to revert to the subject. It is as follows:

In the first place, w, the weight in respect to the selected variable, and j, the order in the coefficients, are terms well understood and need no change or further illustration; i, the degree of the parent quantic, I shall hereafter call the *rank* of the type, $ij - 2w$ which becomes the degree of a covariant got by expanding the differentiant of type $[w: i, j]$ may be called the *grade*. The order and rank may be termed collectively the *permutable indices*.

When a differentiant is given algebraically its weight and order are given but *not* its rank; in addition to the weight and order a third number which may be called the *range* (and which I shall denote by a Greek ε) is

* Writing the type under the form $[2n+1: 2n+1+\nu, 3]$. the degree of the corresponding covariant in the variables is $2n+1+3\nu$, which is the degree in x of the symmetrical function assumed in the text; also each letter in this function occurs 3 times agreeing with the order 3 of the type, and the number of factors in the coefficient of the highest power of x is $2n+1$, which is right for the weight. It is obvious also by inspection that the product $a_1.a_2.\ldots a_{2n+1}$ will arise from each term of the assumed symbolical function affected always with the same sign, so that ε_{2n+1} will occur (as required) in its expression in terms of the coefficients. Of course all the same conclusions will apply if in the formula $(a_1 - a_2)^2(a_3 - a_4)^2 \ldots (a_{2n-1} - a_{2n})^2$ is substituted in lieu of $(a_1 - a_2)(a_2 - a_3)\ldots(a_{2n-1} - a_{2n})(a_{2n} - a_1)$.

That the type to which Q_{2n+1} belongs is non-monadelphic from and after $2n+1 = 9$ is obvious from the fact that that type, when the degree of the parent quantic is made a minimum, is of the form $[2n+1: 2n+1, 3]$, the multiplicity of which is the same as that of $[2n+1: 3, 2n+1]$, or set out in full $[2n+1: 3, 2n+1: 2n+1]$; but cubics include covariants of orders and degrees $2:2$ and $3:3$ among their fundamental forms, and $9:9$ can be formed either by taking a triplication of $3:3$, or by combining $3:3$ with a triplication of $2:2$, so that when $2n+1 = 9$ the type is diadelphic, and *a fortiori*, it is non-monadelphic for values of $2n+1$ superior to 9.

given, being the number less 1 of the letters which enter into it. The relation between *rank* and *range* is one of inequality. The former may be equal to, or greater than, but not less than the latter.

The multiplicity of the type to which a given differentiant belongs is a function of the *weight*, *order* and *rank* and is consequently not known until the *rank* is assigned. Thus, ex. gr. $(ac - b^2)^2$, considered as having the lowest possible rank, viz. 2, (the *range*) is monadelphic; its type is then [2 : 2, 4], but if the rank 4 be assigned to it so that its type is [2.: 4, 4], it becomes diadelphic. We have then, in general, 6 characters (not all independent) appertaining to a differentiant, viz., *weight*, *rank*, *order*, *grade*, *range* and *multiplicity*. The theory of types has never hitherto formed the subject of distinct contemplation, and that is why the necessity for the use of some of the above terms has not been previously felt. But it will have been observed that throughout the preceding memoir it has forced itself upon our notice, and in particular, that it is impossible to go to the bottom of the so-called law of reciprocity or that of the radical representation of forms without keeping in view the question of type and multiplicity.

I have also to remark that since the preceding matter was completed I have been surprised to learn that recent chemical research favors the notion of simple elements (hydrogen atoms in special) being distinguishable from each other in chemical composition. If this view is confirmed, the discrepancy, which I have pointed to, between the known conditions for the existence of algebraical graphs and the unknown natural laws which govern the production of chemical substances may become partially or wholly obliterated, so that, for example, the hydrogen molecule and the extended derivatives from marsh gas may exist in accordance with, and not in contradiction to, algebraical law, and thus it is possible to conceive that all the phenomena of chemistry and algebra may ultimately be shown to be identical.

Since the above matter was sent to press I have been led to study algebraically what may be termed the direct problem of isomerism, that is to say the determination of the number of combinations subject to given conditions that can be formed between the constituents of groups each containing a given number of equivalent chemical atoms, the valences of the several groups being either independent or given linear functions of a certain number of independent parameters. In this problem the numbers of atoms are given and the valences left indeterminate. In the inverse problem the valences are given and the numbers left indeterminate.

The problem of the enumeration of the saturated hydro-carbons, investigated by Professor Cayley, is a simple example of the inverse problem. The direct problem admits of a uniform and unfailing method of solution by generating functions, the exposition of which may probably form the subject of an additional Appendix in the following number.* This method is substantially the same as that which I have described in general terms in the *Comptes Rendus* as applicable to the theory of ternary and other higher varieties of quantics but less difficult of application to the Isomeric Problem on account of the greater simplicity of the crude forms † subject to reduction, which appear in it. Appendix 4 will contain the application of the theory of "Associirter Formen" to the algebraical deduction of the irreducible forms of the quintic and certain other cases which but for the press of matter awaiting publication in the Journal would have formed part (as announced) of the present Appendix.

As already stated in a previous foot-note, the theory of irreducible forms reappears in the isomeric investigation, the general character of the reduced generating function to be interpreted in it being precisely the same as in the invariantive theory, which constitutes an additional and a closer and more real bond of connexion between the chemical and algebraical theories than any which I had in view when I commenced the subject of this memoir.

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* The principle employed in this method leads to the following theorem only a particular case of which comes into play in the general partition problem which covers the ground occupied by the allied invariantive and isomeric theories. Let there be given a product of a limited number of rational functions of

$$u_1^{a_1}.u_2^{a_2}.\ldots.u_i^{a_i};\ u_1^{a'_1}.u_2^{a'_2}.\ldots.u_i^{a'_i};\ \text{etc., etc.,}$$

where all the indices are *positive or negative* integers, and let $\mu_1, \mu_2, \ldots \mu_2$ be given linear functions of $\nu_1, \nu_2, \ldots \nu_j$ (j being not greater than i), then it is always possible to find a limited product of rational functions of

$$v_1^{\beta_1}.v_2^{\beta_2}.\ldots.v_j^{\beta_j};\ v_1^{\beta'_1}.v_2^{\beta'_2}.\ldots.v_j^{\beta'_j};\ \text{etc., etc.,}$$

where the indices are exclusively *positive*, such that the coefficient of $v_1^{\nu_1}.v_2^{\nu_2}\ldots v_j^{\nu_j}$, in their product developed according to ascending powers of $v_1, v_2, \ldots v_j$, shall be the same as the coefficient of $u_1^{\mu_1}u_2^{\mu_2}\ldots.u_i^{\mu_i}$ in the original product developed according to ascending powers of $u_1, u_2, \ldots. u_i$. Previous to the discovery of this principle the problem of isomerism, now completely solved potentially for the direct case, must have remained unattackable by any existing methods, such for example as were known to Euler, the inventor of the application of the method of generating functions to the theory of partitions. It renders supererogatory a large part of the methods devised by myself for the treatment of the problem of compound partitions contained in the printed notes of my lectures on Partitions, delivered at King's College, London, in the year 1859. As an example of the direct problem of isomerism, suppose that three atoms of the same valence j are to combine with ε atoms of hydrogen which do not combine *inter se;* then the number of combinations which can be so formed is the coefficient of $a^j x^\varepsilon$ in the development of the generating function $\dfrac{1+ax+a^2x^2}{(1-a^2)(1-ax)^2(1-ax^3)}$ if the three atoms are all unlike, and of the generating function $\dfrac{1}{(1-a^2)(1-ax)(1-a^2x^2)(1-ax^3)}$ if they are all alike

† Very unluckily printed as "*formes cubiques*" in the Comptes Rendus.

NOTE ON THE LADENBURG CARBON-GRAPH.

The reasoning by which I have established, in the preceding number of the *Journal*, the validity of the Ladenburg graph (and the invalidity of the Kekulean one) as a representative of the root differentiant to a covariant of the 6th degree in the variables and of the 6th order in the coefficients to a quartic, is so peculiar and it may seem to some of my readers so far-fetched, that it appears highly desirable to confirm it by a direct demonstration founded on the principle, that the permutation-sum of the product of the bonds in a valid graph interpreted as differences between the letters which they connect, shall not vanish. Previous to applying this principle to Ladenburg's graph we must convert it into an invariant by attaching hydrogen atoms to the six apices. Let these apices be called a, b, c, d, e, f, and the hydrogen atoms $\alpha, \beta, \gamma, \delta, \varepsilon, \phi$: then the permutation-sum under consideration is

$$\Sigma\,(a-b)(a-c)(b-c)(d-e)(d-f)(e-f)\,(a-d)\,(b-e)\,(e-f)\,(a-\alpha)\,(b-\beta)\,(c-\gamma)\,(d-\delta)\,(e-\varepsilon)\,(f-\phi)$$

where the 6 letters a, b, c, d, e, f are interpermutable, as are also the 6 letters $\alpha, \beta, \gamma, \delta, \varepsilon, \phi$.

It may be well to observe at this point that if we struck off the hydrogen atoms and treated the graph as representing an invariant to a cubic form, the permutation-sum

$$\Sigma\,(a-b)\,(a-c)\,(b-c)\,(d-e)\,(d-f)\,(e-f)\,(a-d)\,(d-c)\,(c-f)$$

would be found to vanish, as may easily be shown and as it ought to do, because there exists no invariant of the 6th order in the coefficients to a cubic form. Let a and d be interchanged in the term given under the sign of summation in the permutation-sum formed from the Ladenburg graph; then the sum of this together with the original term becomes

$$(a-d)\,(b-e)\,(c-f)\,(b-c)\,(e-f)\,(b-\beta)\,(c-\gamma)\,(e-\varepsilon)\,(f-\phi)$$

multiplied by

$$(a\delta-d\alpha)\,(a^2-\overline{b+c}\,a+bc)\,(d^2-\overline{e+f}\,d+ef)-(d\delta-a\alpha)\,(d^2-\overline{b+c}\,d+bc)\,(a^2-\overline{e+f}\,a+ef),$$

which last named multiplier will be found to contain the quantity $(\alpha^3d^2-a^2d^3)\,(\alpha+\delta)$. Again, in the multiplicand, let b and c be interchanged; then, since

$$(b-e)\,(c-f)-(c-e)\,(b-f)=(b-c)\,(e-f),$$

the sum of the original and permuted multiplicand will contain a term

$$(a-d)\,(b-c)^2\,(e-f)^2\,bc\,(e-\varepsilon)\,(f-\phi),$$

and accordingly the entire permutation-sum will contain the terms

$$(\alpha+\delta)\,(a-d)\,(a^3d^2-a^2d^3)\,(b-c)^2(e-f)^2\,bc\Sigma\,(e-\varepsilon)\,(f-\phi).$$

The partial sum last written is

$$4ef+4\varepsilon\phi-2\,(e+f)\,(\varepsilon+\phi).$$

Hence we may readily see that the total permutation-sum will contain *inter alia* a positive multiple of the combination $\alpha^4b^3e^3d^2cf\alpha$ and will not vanish, and consequently the graph is valid and not illusory; I presume that the same method applied to Kekulé's graph regarded as a representation of the covariant to the type [9 : 4, 6 : 6], which is the same thing (except that the hydrogen atoms are suppressed) as the graph to the invariant [15 : 4, 6 ; 1, 6 : 0], would serve to show it to be illusory as previously inferred from other considerations.

ERRATA IN THE PART OF THIS MEMOIR INCLUDED IN THE PRECEDING NUMBER OF THE JOURNAL, pp. 64–104.

Page 66. line 4, *for* quartic *read* quadric.
Page 66, line 4, (from foot) *for* irrefragible *read* irrefragable.
Page 70, line 17, *after* covariant *insert* of.
Page 74, line 6, *for* the second *read* Fig. 15.
Page 74, lines 25, 29, *for* Fig. 43 *read* Fig. 45.
Page 76, line 11, *for* pluralty *read* plurality.
Page 77, line 5, (from foot), *for* *CA* *read* *CB*.
Page 79, last paragraph, *for* x *read* y.
Page 80, line 10, *for* a_2d *read* a^2d.
Page 86, line 26, *for* enparametric *read* henparametric.
Page 87, line 12, *for* expressions *read* expression.
Page 91, line 15, *for* j *read* i.
Page 93, line 12, *for* *abe* *read* *ace*.
Page 93, line 16, *for* multiplier *read* multiple.
Page 94, line 2, *for* $-\frac{1}{16}(\alpha\beta$ *read* $-\frac{1}{216}(\alpha\beta$.
Page 97, line 5, (from foot,) *for* a $\lambda:\mu:\nu$ *read* as $\lambda:\mu:\nu$.
Page 98, line 9, *for* $I^2U_2=a'U_1+b'U_2+c'U_3$ *read* $I^2U_1.=aU_1+bU_2+cU_3$.
Page 98, line 18, *for* which makes *read* which make.
Page 99, line 5, *for* none *read* no.
Page 99, line 19, *for* $\lambda U+\mu V$ *read* $\Lambda U+\mathrm{M}V$.
Page 101, line 6, *dele period at end of line.*
Page 101, line 20, *after* biologists *insert* term.
Page 101, line 3, (from foot,) *for* y_r *read* y^r.
Page 102, middle of page, *all the indices of* ϵ *should be subscript.*
Page 102, foot-note, *for* $q=1$ *read* $q=i$.
Page 104, last line of text, *for* conclusiou *read* conclusion.